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Complex oscillation of meromorphic solutions for difference Riccati equation

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Abstract
In this paper, we investigate zeros and α-points of meromorphic solutions $f(z)$ for difference Riccati equations, and we obtain some estimates of exponents of convergence of zeros and α-points of $f(z)$ and shifts $f(z+n)$, differences $\Delta f(z) = f(z+1) - f(z)$, and divided differences $\frac{\Delta f(z)}{f(z)}$.

MSC: 30D35; 39B12

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1 Introduction and main results

In this paper, we assume that the reader is familiar with the standard notations and basic results of Nevanlinna's value distribution theory (see [1, 2]). In addition, we use the notions $\sigma(f)$ to denote the order of growth of the meromorphic function $f(z)$, $\lambda(f)$, and $\lambda(\frac{1}{f})$ to denote the exponents of convergence of zeros and poles of $f(z)$, respectively. We say a meromorphic function $f(z)$ is oscillatory if $f(z)$ has infinitely many zeros.

The theory of difference equations, the methods used in their solutions, and their wide applications have advanced beyond their adolescent stage to occupy a central position in applicable analysis. The theory of oscillation play an important role in the research on discrete equations, and it is systematically introduced in [3]. The complex oscillation is the development and deepening of the corresponding real oscillation, and it can profoundly reveals the essence of the oscillation problem that the property of oscillation is investigated in complex domain.

Recently, as the difference analogs of Nevanlinna's theory were being investigated [4–6], many results on the complex difference equations have been got rapidly. Many papers [4, 7–9] mainly deal with the growth of meromorphic solutions of some difference equations, and several papers [7, 8, 10–15] deal with analytic properties of meromorphic solutions of some nonlinear difference equations. Especially, there has been an increasing interest in studying difference Riccati equations in the complex plane [8, 10, 12, 15].

In [8], Ishizaki gave some surveys of the basic properties of the difference Riccati equation

$$y(z+1) = \frac{A(z) + y(z)}{1 - y(z)},$$

where $A(z)$ is a rational function, which have analogs in the differential case [16]. In the proof of the celebrated classification theorem, Halburd and Korhonen [13] were concerned with the difference Riccati equation of the form

$$w(z+1) = \frac{A(z) + \delta w(z)}{\delta - w(z)},$$

where A is a polynomial, $\delta = \pm 1$. In [10], Chen and Shon investigated the existence and forms of rational solutions, and the Borel exceptional value, zeros, poles, and fixed points of transcendental solutions, and they proved the following theorem.

Theorem A *Let $\delta = \pm 1$ be a constant and $A(z) = \frac{m(z)}{n(z)}$ be an irreducible nonconstant rational function, where $m(z)$ and $n(z)$ are polynomials with $\deg m(z) = m$ and $\deg n(z) = n$.*

If $f(z)$ is a transcendental finite order meromorphic solution of the difference Riccati equation

$$f(z+1) = \frac{A(z) + \delta f(z)}{\delta - f(z)}, \tag{1}$$

then

(i) *if $\sigma(f) > 0$, then $f(z)$ has at most one Borel exceptional value*;

(ii) $\lambda(\frac{1}{f}) = \lambda(f) = \sigma(f)$;

(iii) *if $A(z) \not\equiv -z^2 - z + 1$, then the exponent of convergence of fixed points of $f(z)$ satisfies $\tau(f) = \sigma(f)$.*

In [15], the first author investigated fixed points of meromorphic functions $f(z)$ for difference Riccati equation (1), and obtain some estimates of exponents of convergence of fixed points of $f(z)$ and shifts $f(z+n)$, differences $\Delta f(z) = f(z+1) - f(z)$, and divided differences $\frac{\Delta f(z)}{f(z)}$.

In this paper, we investigate zeros and α-points of meromorphic solutions $f(z)$ for difference Riccati equations (1), and we obtain some estimates of the exponents of convergence of zeros and α-points of $f(z)$ and shifts $f(z+n)$, differences $\Delta f(z) = f(z+1) - f(z)$, and divided differences $\frac{\Delta f(z)}{f(z)}$ of meromorphic solutions of (1). We prove the following theorem.

Theorem 1.1 *Let $\delta = \pm 1$ be a constant and $A(z)$ be a nonconstant rational function. Set $\Delta f(z) = f(z+1) - f(z)$. If there exists a nonconstant rational function $s(z)$ such that $A(z) = -s^2(z)$, then every finite order transcendental meromorphic solution $f(z)$ of the difference Riccati equation (1), its difference $\Delta f(z)$, and divided difference $\frac{\Delta f(z)}{f(z)}$ are oscillatory and satisfy*

$$\lambda\big(\Delta f(z)\big) = \lambda\left(\frac{\Delta f(z)}{f(z)}\right) = \sigma(f).$$

Theorem 1.2 *Let $A(z)$ be a nonconstant rational function. If α is a non-zero complex constant, then every finite order transcendental meromorphic solution $f(z)$ of the difference Riccati equation*

$$f(z+1) = \frac{A(z) + f(z)}{1 - f(z)} \tag{2}$$

satisfies

(i) *if* $\alpha \neq -1$, *then* $\lambda(f(z+n)-\alpha) = \sigma(f)$, $n = 0,1,2,\ldots$;
(ii) *if there is a rational function* $n(z)$ *satisfying*

$$A(z) = \frac{\alpha^2}{4(1+\alpha)} - (1+\alpha)n^2(z),$$

then $\lambda(\frac{\Delta f(z)}{f(z)} - \alpha) = \sigma(f)$;
(iii) *if there is a rational function* $m(z)$ *satisfying*

$$A(z) = \frac{\alpha^2+\alpha}{4} - m^2(z),$$

then $\lambda(\Delta f(z) - \alpha) = \sigma(f)$.

Example 1.1 The function $f(z) = \frac{Q(z)-2z(z-1)(z+1)}{zQ(z)+z^2(z-1)(z+1)}$ satisfies the difference Riccati equation

$$f(z+1) = \frac{A(z)+f(z)}{1-f(z)},$$

where $A(z) = -\frac{2}{z(z+1)}$, $Q(z)$ is a periodic function with period 1. Note that for any $\rho \in [1,+\infty)$, there exists a prime periodic entire function $Q(z)$ of order $\sigma(Q) = \rho$ by Ozawa [17]. Thus $\sigma(f) = \sigma(Q) = \rho \geq 1$.

Also, this solution $f(z) = \frac{Q(z)-2z(z-1)(z+1)}{zQ(z)+z^2(z-1)(z+1)}$ satisfies

$$\Delta f(z) = f(z+1) - f(z) = \frac{18z^3(z+1)^3 - [Q(z)-2z(z+1)(2z+1)]^2}{z(z+1)[Q(z)+z(z-1)(z+1)][Q(z)+z(z+1)(z+2)]}$$

and

$$\frac{\Delta f(z)}{f(z)} = \frac{18z^3(z+1)^3 - [Q(z)-2z(z+1)(2z+1)]^2}{(z+1)[Q(z)-2z(z-1)(z+1)][Q(z)+z(z+1)(z+2)]}.$$

Using the same discussion as Lemma 2.1, we easily see that $18z^3(z+1)^3 - [Q(z)-2z(z+1)(2z+1)]^2$ and $[Q(z)+z(z-1)(z+1)][Q(z)+z(z+1)(z+2)]$ (or $[Q(z)-2z(z-1)(z+1)][Q(z)+z(z+1)(z+2)]$) have at most finitely many common zeros. Thus,

$$\begin{aligned}\lambda(\Delta f(z)) &= \lambda\left(\frac{\Delta f(z)}{f(z)}\right) = \lambda\left(18z^3(z+1)^3 - \left[Q(z)-2z(z+1)(2z+1)\right]^2\right)\\ &= \sigma(Q) = \sigma(f) = \rho \geq 1.\end{aligned}$$

2 Lemmas for proofs of theorems

Firstly we need the following lemmas for the proof of Theorem 1.1.

Lemma 2.1 *Let* $A(z)$ *be a nonconstant rational function, and* $f(z)$ *be a nonconstant meromorphic function. Then*

$$y_1(z) = A(z) + f^2(z) \quad \textit{and} \quad y_2(z) = 1 - f(z)$$

have at most finitely many common zeros.

Proof Suppose that z_0 is a common zero of $y_1(z)$ and $y_2(z)$. Then $y_2(z_0) = 1 - f(z_0) = 0$. Thus, $f(z_0) = 1$. Substituting $f(z_0) = 1$ into $y_1(z)$, we obtain

$$y_1(z_0) = A(z_0) + 1 = 0.$$

Since $A(z)$ is a nonconstant rational function, $A(z) + 1$ has only finitely many zeros. Thus, $y_1(z)$ and $y_2(z)$ have at most finitely many common zeros. □

Lemma 2.2 *Let $w(z)$ be a nonconstant finite order transcendental meromorphic solution of the difference equation of*

$$P(z, w) = 0,$$

where $P(z, w)$ is a difference polynomial in $w(z)$. If $P(z, \alpha) \not\equiv 0$ for a meromorphic function $\alpha(z)$ satisfying $T(r, \alpha) = S(r, w)$, then

$$m\left(r, \frac{1}{w - \alpha}\right) = S(r, w)$$

holds for all r outside of a possible exceptional set with finite logarithmic measure.

3 Proof of Theorem 1.1

Suppose that $\delta = 1$. We only prove the case $\delta = 1$. We can use the same method to prove the case $\delta = -1$.

First, we prove that $\lambda(\Delta f(z)) = \sigma(f(z))$.

By (1) and the fact that $A(z) = -s^2(z)$, we obtain

$$\begin{aligned} \Delta f(z) &= f(z+1) - f(z) = \frac{A(z) + f(z)}{1 - f(z)} - f(z) = \frac{A(z) + f^2(z)}{1 - f(z)} \\ &= \frac{f^2(z) - s^2(z)}{1 - f(z)} = \frac{[f(z) - s(z)][f(z) + s(z)]}{1 - f(z)}. \end{aligned} \tag{3}$$

Since $A(z)$ and $s(z)$ are rational functions, we know that $f(z) - s(z)$ (or $f(z) + s(z)$) and $1 - f(z)$ have the same poles, except possibly finitely many. By Lemma 2.1, we see that $A(z) + f^2(z)$ and $1 - f(z)$ have at most finitely many common zeros. Hence, by (3), we only need to prove that

$$\lambda\big(f(z) - s(z)\big) = \sigma\big(f(z)\big) \quad \text{or} \quad \lambda\big(f(z) + s(z)\big) = \sigma\big(f(z)\big). \tag{4}$$

Suppose that $\lambda(f(z) - s(z)) < \sigma(f(z))$. By $\sigma(f(z) - s(z)) = \sigma(f(z))$ and Hadamard factorization theorem, $f(z) - s(z)$ can be rewritten in the form

$$f(z) - s(z) = z^t \frac{P_0(z)}{Q_0(z)} e^{h(z)} = \frac{P(z)}{Q(z)}, \tag{5}$$

where $h(z)$ is a polynomial with $\deg h(z) \le \sigma(f(z))$, $P_0(z)$ and $Q_0(z)$ are canonical products ($P_0(z)$ may be a polynomial) formed by non-zero zeros and poles of $f(z) - s(z)$, respectively, t is an integer, if $t \ge 0$, then $P(z) = z^t P_0(z)$, $Q(z) = Q_0(z)e^{-h(z)}$; if $t < 0$, then $P(z) = P_0(z)$,

$Q(z) = z^{-t}Q_0(z)e^{-h(z)}$. Combining Theorem A with the property of the canonical product, we have

$$\begin{cases} \sigma(P(z)) = \lambda(P(z)) = \lambda(f(z) - s(z)) < \sigma(f(z)), \\ \sigma(Q(z)) = \lambda(Q(z)) = \sigma(f(z)). \end{cases} \tag{6}$$

By (5), we obtain

$$f(z) = s(z) + P(z)y(z), \qquad f(z+1) = s(z+1) + P(z+1)y(z+1), \tag{7}$$

where $y(z) = \frac{1}{Q(z)}$. Thus, by (6), we have

$$\sigma\big(y(z)\big) = \sigma\big(Q(z)\big) = \sigma\big(f(z)\big), \qquad \sigma\big(P(z+1)\big) = \sigma\big(P(z)\big) < \sigma\big(f(z)\big).$$

Substituting (7) into (1), we obtain

$$\begin{aligned} E_1(z,y) := {} & \big[s(z+1) + P(z+1)y(z+1)\big]\big[1 - s(z) - P(z)y(z)\big] \\ & - A(z) - s(z) - P(z)y(z) = 0. \end{aligned} \tag{8}$$

By (8) and the fact that $A(z) = -s^2(z)$, we have

$$\begin{aligned} E_1(z,0) := {} & s(z+1)\big[1 - s(z)\big] - A(z) - s(z) \\ = {} & s(z+1)\big[1 - s(z)\big] + s^2(z) - s(z) \\ = {} & \big[1 - s(z)\big]\big[s(z+1) - s(z)\big]. \end{aligned}$$

Since $s(z)$ is a nonconstant rational faction, we see that $1 - s(z) \not\equiv 0$ and $s(z+1) - s(z) \not\equiv 0$, so that

$$E_1(z,0) \not\equiv 0. \tag{9}$$

Thus, by (6), (9), and Lemma 2.2, we obtain for any given ε $(0 < \varepsilon < \sigma(f(z)) - \sigma(P(z)))$,

$$N\left(r, \frac{1}{y(z)}\right) = T\big(r, y(z)\big) + S\big(r, y(z)\big) + O\big(r^{\sigma(P(z))+\varepsilon}\big) \tag{10}$$

holds for all r outside of a possible exceptional set with finite logarithmic measure.

On the other hand, by $y(z) = \frac{1}{Q(z)}$ and the fact that $Q(z)$ is an entire function, we see that

$$N\left(r, \frac{1}{y(z)}\right) = N\big(r, Q(z)\big) = 0. \tag{11}$$

Thus (10) is a contradiction. Hence, (4) holds, that is, $\lambda(\Delta f(z)) = \sigma(f(z))$.

Secondly, we prove that $\lambda(\frac{\Delta f(z)}{f(z)}) = \sigma(f)$. By (1), we obtain

$$\frac{\Delta f(z)}{f(z)} = \frac{[f(z) - s(z)][f(z) + s(z)]}{f(z)(1 - f(z))}.$$

Thus, by this and (4), we see that $\lambda(\frac{\Delta f(z)}{f(z)}) = \sigma(f)$.

4 Proof of Theorem 1.2

Suppose that $f(z)$ is a finite order transcendental meromorphic solution of (2).

(i) First, we prove that the conclusion holds when $n = 0$. Set $y(z) = f(z) - \alpha$. Thus, $y(z)$ is transcendental, $T(r,y) = T(r,f) + O(\log r)$, and $S(r,y) = S(r,f)$. Substituting $f(z) = y(z) + \alpha$ into (2), we obtain

$$K_0(z,y) = \bigl[y(z+1) + \alpha\bigr]\bigl[1 - y(z) - \alpha\bigr] - A(z) - y(z) - \alpha = 0.$$

Thus

$$K_0(z,0) = \alpha(1-\alpha) - A(z) - \alpha = -\alpha^2 - A(z).$$

By the condition that $A(z)$ is a nonconstant rational function, we obtain $K_0(z,0) \not\equiv 0$. By Lemma 2.2,

$$N\left(r, \frac{1}{y}\right) = T(r,y) + S(r,y)$$

holds for all r outside of a possible exceptional set with finite logarithmic measure. That is,

$$N\left(r, \frac{1}{f-\alpha}\right) = T(r,f) + S(r,f)$$

holds for all r outside of a possible exceptional set with finite logarithmic measure. Thus, we obtain $\lambda(f(z) - \alpha) = \sigma(f(z))$.

Now suppose that $n = 1$. By (2) and $\alpha \neq -1$, we see that

$$\begin{aligned} f(z+1) - \alpha &= \frac{A(z) + f(z)}{1 - f(z)} - \alpha = \frac{(1+\alpha)f(z) + A(z) - \alpha}{1 - f(z)} \\ &= (1+\alpha) \cdot \frac{f(z) + \frac{A(z)-\alpha}{1+\alpha}}{1 - f(z)}. \end{aligned} \tag{12}$$

Using the same discussion as Lemma 2.1, we easily see that $f(z) + \frac{A(z)-\alpha}{1+\alpha}$ and $1 - f(z)$ have at most finitely many common zeros. Thus, we only need to prove that

$$\lambda\left(f(z) + \frac{A(z) - \alpha}{1+\alpha}\right) = \sigma(f). \tag{13}$$

Using the same method as in the proof of (4)-(11), we can prove that (13) holds. Hence $\lambda(f(z+1) - \alpha) = \sigma(f(z))$.

Now in (12), we replace z by $z + n - 1$ ($n \geq 1$), and we obtain

$$f(z+n) - \alpha = (1+\alpha) \cdot \frac{f(z+n-1) + \frac{A(z+n-1)-\alpha}{1+\alpha}}{1 - f(z+n-1)}. \tag{14}$$

Set $g(z) = f(z+n-1)$. Then (14) is transformed as

$$g(z+1) - \alpha = (1+\alpha) \cdot \frac{g(z) + \frac{A(z+n-1)-\alpha}{1+\alpha}}{1 - g(z)}. \tag{15}$$

Since $A(z+n-1)$ is a nonconstant rational function too, applying the conclusion for $n=1$ to (15), we obtain

$$\lambda\big(f(z+n)-\alpha\big)=\lambda\big(g(z+1)-\alpha\big)=\sigma(g)=\sigma(f),\quad n=2,3,\ldots.$$

(ii) Suppose that there is a rational function $n(z)$ satisfying

$$A(z)=\frac{\alpha^2}{4(1+\alpha)}-(1+\alpha)n^2(z).\tag{16}$$

Now we prove

$$\lambda\left(\frac{\Delta f(z)}{f(z)}-\alpha\right)=\sigma(f).\tag{17}$$

By (2), we have

$$\frac{\Delta f(z)}{f(z)}-\alpha=\frac{f(z+1)-f(z)}{f(z)}-\alpha=\frac{(1+\alpha)f^2(z)-\alpha f(z)+A(z)}{f(z)(1-f(z))}.\tag{18}$$

If $\alpha=-1$, then

$$\frac{\Delta f(z)}{f(z)}-\alpha=\frac{A(z)-\alpha f(z)}{f(z)(1-f(z))}.\tag{19}$$

Since $A(z)$ is a rational function, $A(z)-\alpha f(z)$ and $(1-f(z))$ have the same poles, except possibly finitely many. By (19) and Theorem A, we obtain

$$\lambda\left(\frac{\Delta f(z)}{f(z)}-\alpha\right)=\lambda\left(\frac{A(z)-\alpha f(z)}{f(z)(1-f(z))}\right)=\lambda\left(\frac{1}{f}\right)=\sigma(f).$$

If $\alpha\neq-1$, by (16) and (18), we have

$$\frac{\Delta f(z)}{f(z)}-\alpha=(1+\alpha)\cdot\frac{[f(z)-\frac{\alpha}{2(1+\alpha)}+n(z)][f(z)-\frac{\alpha}{2(1+\alpha)}-n(z)]}{f(z)(1-f(z))}.\tag{20}$$

Using the same discussion as Lemma 2.1, we easily see that $(1+\alpha)f^2(z)-\alpha f(z)+A(z)$ and $f(z)(1-f(z))$ have at most finitely many common zeros. Thus, by (20), in order to prove (17), we only need to prove that

$$\lambda\left(f(z)-\frac{\alpha}{2(1+\alpha)}-n(z)\right)=\sigma\big(f(z)\big)\tag{21}$$

or

$$\lambda\left(f(z)-\frac{\alpha}{2(1+\alpha)}+n(z)\right)=\sigma\big(f(z)\big).$$

Without loss of generality, we prove (21). Suppose that $\lambda(f(z)-\frac{\alpha}{2(1+\alpha)}-n(z))<\sigma(f(z))$. Using the same method as in the proof of (4)-(11), we see that $f(z)-\frac{\alpha}{2(1+\alpha)}-n(z)$ can be rewritten as

$$f(z) = \frac{\alpha}{2(1+\alpha)} + n(z) + P(z)y(z), \tag{22}$$

where $y(z) = \frac{1}{Q(z)}$, $P(z)$, $Q(z)$ are non-zero entire functions, such that

$$\lambda\big(P(z)\big) = \sigma\big(P(z)\big) < \sigma\big(f(z)\big) \quad \text{and} \quad \lambda\big(Q(z)\big) = \sigma\big(Q(z)\big) = \sigma\big(f(z)\big).$$

Substituting (22) into (2), we obtain

$$\begin{aligned} K_1(z,y) := {} & \left[\frac{\alpha}{2(1+\alpha)} + n(z+1) + P(z+1)y(z+1)\right] \\ & \cdot \left[1 - \frac{\alpha}{2(1+\alpha)} - n(z) - P(z)y(z)\right] - A(z) - \frac{\alpha}{2(1+\alpha)} \\ & - n(z) - P(z)y(z) = 0 \end{aligned}$$

and

$$\begin{aligned} K_1(z,0) := {} & \left[\frac{\alpha}{2(1+\alpha)} + n(z+1)\right]\left[1 - \frac{\alpha}{2(1+\alpha)} - n(z)\right] \\ & - A(z) - \frac{\alpha}{2(1+\alpha)} - n(z). \end{aligned}$$

By the above equation and (16), we have

$$\begin{aligned} K_1(z,0) := {} & \left[\frac{\alpha}{2(1+\alpha)} + n(z+1)\right]\left[1 - \frac{\alpha}{2(1+\alpha)} - n(z)\right] \\ & - \frac{\alpha^2}{4(1+\alpha)} + (1+\alpha)n^2(z) - \frac{\alpha}{2(1+\alpha)} - n(z) \\ = {} & (1+\alpha)n^2(z) - n(z)n(z+1) + \frac{2+\alpha}{2(1+\alpha)}n(z+1) \\ & - \frac{2+3\alpha}{2(1+\alpha)}n(z) - \frac{(3+\alpha)\alpha^2}{4(1+\alpha)^2} \\ = {} & R(z) - \frac{(3+\alpha)\alpha^2}{4(1+\alpha)^2}, \end{aligned}$$

where $R(z) = (1+\alpha)n^2(z) - n(z)n(z+1) + \frac{2+\alpha}{2(1+\alpha)}n(z+1) - \frac{2+3\alpha}{2(1+\alpha)}n(z)$. Since $\frac{(3+\alpha)\alpha^2}{4(1+\alpha)^2}$ is a constant, to prove $K_1(z,0) \not\equiv 0$, we need to prove that $R(z)$ is nonconstant.

Now we prove that

$$R(z) = (1+\alpha)n^2(z) - n(z)n(z+1) + \frac{2+\alpha}{2(1+\alpha)}n(z+1) - \frac{2+3\alpha}{2(1+\alpha)}n(z)$$

is nonconstant. Since $A(z)$ is a nonconstant rational function and due to (16), $n(z)$ is a nonconstant rational function too. First, if $n(z)$ is a polynomial with $\deg n(z) = n \geq 1$, then

$$\deg\big((1+\alpha)n^2(z) - n(z)n(z+1)\big) = \deg n(z)\big((1+\alpha)n(z) - n(z+1)\big) = 2n$$

is the maximal degree in $R(z)$ (since $\alpha \neq 0,1$). Thus $R(z)$ is a polynomial with $\deg R(z) = 2n \geq 2$. Secondly, if $n(z) = \frac{p(z)}{q(z)}$, where $p(z)$ and $q(z)$ are polynomials with $\deg p(z) = p < q = \deg q(z)$, then $R(z) = \frac{s(z)}{t(z)}$, where

$$\begin{aligned} s(z) &= (1+\alpha)p^2(z)q(z+1) - p(z)p(z+1)q(z) \\ &\quad + (2+\alpha)p(z+1)q^2(z) - (2+3\alpha)p(z)q(z)q(z+1) \end{aligned}$$

and

$$t(z) = 2(1+\alpha)q^2(z)q(z+1).$$

Since $p < q$,

$$\begin{aligned} \deg s(z) &= \deg\bigl((2+\alpha)p(z+1)q^2(z) - (2+3\alpha)p(z)q(z)q(z+1)\bigr) \\ &= 2q + p < 3q = \deg t(z). \end{aligned}$$

Thus $R(z)$ is nonconstant. Lastly, if $n(z) = \frac{p(z)}{q(z)}$, where $p(z)$ and $q(z)$ are polynomials with $\deg p(z) = p \geq q = \deg p(z)$, then

$$n(z) = n_1(z) + \frac{p_1(z)}{q_1(z)},$$

where $n_1(z)$, $p_1(z)$, and $q_1(z)$ are polynomials with $\deg n_1(z) = p - q \geq 0$ and $\deg p_1(z) < \deg q(z)$. By the above discussion, we know that $R(z)$ is nonconstant. Hence $K_1(z,0) \not\equiv 0$, and, by Lemma 2.2, we see that (21) holds.

(iii) Suppose that there is a rational function $m(z)$ satisfying

$$A(z) = \frac{\alpha^2 + \alpha}{4} - m^2(z). \tag{23}$$

In what follows, we prove that

$$\lambda\bigl(\Delta f(z) - \alpha\bigr) = \sigma(f). \tag{24}$$

By (2) and (23), we obtain

$$\begin{aligned} \Delta f(z) - \alpha &= \frac{A(z) + f^2(z)}{1 - f(z)} - \alpha = \frac{f^2(z) + \alpha f(z) + A(z) - \alpha}{1 - f(z)} \\ &= \frac{[f(z) + \frac{\alpha}{2} + m(z)][f(z) + \frac{\alpha}{2} - m(z)]}{1 - f(z)}. \end{aligned} \tag{25}$$

Using the same discussion as Lemma 2.1, we easily see that $f^2(z) + \alpha f(z) + A(z) - \alpha$ and $1 - f(z)$ have at most finitely many common zeros. Thus, by (25), we know that to prove (24), we only need to prove that

$$\lambda\left(f(z) + \frac{\alpha}{2} + m(z)\right) = \sigma\bigl(f(z)\bigr) \quad \text{or} \quad \lambda\left(f(z) + \frac{\alpha}{2} - m(z)\right) = \sigma\bigl(f(z)\bigr).$$

Using the same method as in the proof of (21), we can prove that the above equation holds.

Thus, Theorem 1.2 is proved.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Y-YJ completed the main part of this article, Y-YJ, Z-QM, and MW corrected the main theorems. All authors read and approved the final manuscript.

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References
1. Hayman, WK: Meromorphic Functions. Clarendon, Oxford (1964)
2. Laine, I: Nevanlinna Theory and Complex Differential Equations. de Gruyter, Berlin (1993)
3. Agarwal, RP, Bohner, M, Grace, SR, O'Regan, D: Discrete Oscillation Theory. Hindawi Publishing Corporation, New York (2005)
4. Chiang, YM, Feng, SJ: On the Nevanlinna characteristic of $f(z+\eta)$ and difference equations in the complex plane. Ramanujan J. **16**, 105-129 (2008)
5. Halburd, RG, Korhonen, R: Difference analogue of the lemma on the logarithmic derivative with applications to difference equations. J. Math. Anal. Appl. **314**, 477-487 (2006)
6. Halburd, RG, Korhonen, R: Nevanlinna theory for the difference operator. Ann. Acad. Sci. Fenn., Math. **31**, 463-478 (2006)
7. Chen, ZX: On growth, zeros and poles of meromorphic functions of linear and nonlinear difference equations. Sci. China Ser. A **54**, 2123-2133 (2011)
8. Ishizaki, K: On difference Riccati equations and second order linear difference equations. Aequ. Math. **81**, 185-198 (2011)
9. Laine, I, Yang, CC: Clunie theorems for difference and q-difference polynomials. J. Lond. Math. Soc. **76**, 556-566 (2007)
10. Chen, ZX, Shon, KH: Some results on Riccati equations. Acta Math. Sin. **27**, 1091-1100 (2011)
11. Chen, ZX, Huang, Z, Zhang, R: On difference equations relating to gamma function. Acta Math. Sin. **31**, 1281-1294 (2011)
12. Halburd, RG, Korhonen, R: Existence of finite-order meromorphic solutions as a detector of integrability in difference equations. Physica D **218**, 191-203 (2006)
13. Halburd, RG, Korhonen, R: Finite-order meromorphic solutions and the discrete Painlevé equations. Proc. Lond. Math. Soc. **94**, 443-474 (2007)
14. Jiang, YY, Chen, ZX: Value distribution of meromorphic solutions to some difference equations. J. South China Norm. Univ., Nat. Sci. Ed. **45**(1), 19-23 (2013)
15. Jiang, YY, Chen, ZX: Fixed points of meromorphic solutions for difference Riccati equation. Taiwan. J. Math. **17**(4), 1413-1423 (2013)
16. Bank, SB, Gundersen, G, Laine, I: Meromorphic solutions of the Riccati differential equation. Ann. Acad. Sci. Fenn., Ser. A I Math. **6**(2), 369-398 (1982)
17. Ozawa, M: On the existence of prime periodic entire functions. Kodai Math. Semin. Rep. **29**, 308-321 (1978)

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